International Journal of Theoretical Physics, Vol. 32, No. 9, 1993

Theory of Fiber Bundle with Local Metric of Internal Space and Gravitation with Torsion

Li Yuanjie[1]

Received December 11, 1991

In this paper we propose a new theory of a fiber bundle provided with a local metric of internal space. The fibers differ from usual fibers, having an enlarged factor. The enlargement may be procured by a differential mapping $\phi(x)$ from structure group G to the fiber F_x at $x \in M$, and $\phi(x) \in R$. The torsion presented stems from the local metric of internal space and the local metric stems from a induced mapping $\phi_*(x)$ of $\phi(x)$. From the theory we can get the Brans–Dicke theory with torsion. If we assume the spin density of the gauge field determines the enlarged factor of the fiber F_x, our theory is an extended Cartan theory.

1. INTRODUCTION

The gauge theory is related to the profound concept of the mathematical theory of fiber bundles (Wu and Yang, 1975, 1976). It was pointed out first by Utiyama (1956) and then by a number of physicists (e.g., Sciama, 1962) that gravitation theory can be looked upon as a non-Abelian gauge theory, and it becomes possible to develop and generalize Einstein's theory of gravity. In recent years, on account of the development in superstring theories, higher-dimensional theories have been paid more and more attention. Cho (1975) proposed a higher-dimensional unification of gravitation and gauge theory. Madore (1990) presented a modification of the traditional Kaluza–Klein theory by a noncommutative geometry based on a semisimple algebra. In his article the union of space-time and an internal space was described by the principal bundle.

The Einstein–Cartan equations with torsion were also obtained from some gauge theories (Kibble, 1961). Kalinowski (1981) generalized the Cartan torsion to higher-dimensional Kaluza–Klein theory.

[1]Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, China.

Nevertheless, what is the geometric cause producing torsion in these theories and what is the physical cause? These questions so far have had no satisfying answer. The purpose of this paper is to find the geometric and physical causes.

2. CHOICE OF THE BASIS IN THE PRINCIPAL FIBER BUNDLE

We introduce a $(4+n)$-dimensional Riemann–Cartan space; its quotient space by the equivalence relation of group transformations is the usual 4-dimensional space-time, and the enlarged space is a principal fiber bundle space. We denote the principal fiber bundle as $P(M, G)$, where M is the base manifold and G the structural group. Now we define a differential mapping $\phi(x)$ from G to the fiber F_x at $x \in M$,

$$\phi(x): \quad a \in G \to p(x, \phi(x)\, a) \in F_x \tag{1}$$

and $\phi(x) \in R$. Therefore we get a new principal fiber bundle $P(M, F)$ by the differential mapping $\phi(x)$.

First let us choose a coordinate basis $\xi_\mu = \partial_\mu$ for the base manifold M, whose commutation relations are trivial,

$$[\xi_\mu, \xi_\nu] = [\partial_\mu, \partial_\nu] = 0$$

For a basis of G, choose a set of n linearly independent left invariant vector fields ξ_i $(i = 5 \to 4+n)$ on G. These ξ_i can also be viewed as a basis of the Lie algebra $\mathscr{G}$ of G. The commutation relations of these vector fields are

$$[\xi_i, \xi_j] = f^k_{ij} \xi_k$$

where the f^k_{ij} are structural constants of G.

Next we shift $\xi_\mu \in T_x(M)$ and $\xi_i \in \mathscr{G}$ to the tangent space of the principal fiber bundle at $p \in P$ by the induced mappings, respectively. Notice that the $\mathscr{G}$-valued connection 1-form ω is defined as

$$\omega: \quad \xi^*_i \in \mathscr{G}_p(F_x) \to \xi_i \in \mathscr{G} \tag{2}$$

Using the induced mapping $\phi_*(x)$ of $\phi(x)$ and the converse mapping of ω, we can shift $\xi_i \in \mathscr{G}$ to $T_p(F_x)$,

$$\phi_*(x)\, \omega^{-1}: \quad \xi_i \in \mathscr{G} \to \hat{\xi}_i = \phi_*(x)\, \xi^*_i \in T_p(F_x) \tag{3}$$

Because $\mathscr{G}_p(F_x)$ is an isomorphic Lie algebra of G, $\mathscr{G}_p(F_x)$ is denoted as the Lie algebra of the tangent space at $p \in F_x$. Therefore

$$[\xi^*_i, \xi^*_j] = f^k_{ij} \xi^*_k$$

and we have

$$[\hat{\xi}_i, \hat{\xi}_j] = \phi_*(x) f_{ij}^k \hat{\xi}_k = F_{ij}^k(x) \hat{\xi}_k \tag{4}$$

where $F_{ij}^k(x)$ are still constants at the fixed point x, $F_{ij}^k(x) = \phi_*(x) f_{ij}^k$. The shifting of ξ_μ can be procured by the usual method called a horizontal lift. We know that the tangent space $T_p(P)$ of the principal fiber bundle at $p \in P$ is the direct sum of V_p and H_p, where we denote V_p as the space of vertical vectors that is tangent to the fiber F_x at $p \in F_x$, and H_p as the space of horizontal vectors. Notice that the fiber over x is defined as $\pi^{-1}(x) = F_x$ also,

$$\pi_p^{-1}: \quad x \in M \rightarrow \pi_p^{-1}(x) \in F_x$$

The induced mapping of π_p^{-1} is

$$d\pi_p^{-1}: \quad \xi_\mu \in T_x(M) \rightarrow \hat{\xi}_\mu = d\pi^{-1} \xi_\mu \in H_p \tag{5}$$

From the induced mapping $d\pi_p^{-1}$, we give a connection $B(P)$ over the principal fiber bundle

$$\hat{\xi}_\mu = D_\mu = \partial_\mu - B_\mu^i \hat{\xi}_i \in H_p \tag{6}$$

Since the space $T_p(F_x)$ is tangent to F_x, we have $V_p = T_p(F_x)$.

Through the above discussion, we choose $(\hat{\xi}_i, \hat{\xi}_\mu)$ as a basis set of the principal fiber bundle at $p \in P$.

3. CHOICE OF A METRIC

We assume that M is a metric manifold with the metric $g_{\mu\nu}$; for internal space we choose the metric I_{ij},

$$I_{ij}(x) = F_{in}^k(x) F_{kj}^n(x) = \phi_*^2(x) \eta_{ij} \tag{7}$$

where η_{ij} is the Minkowski metric; evidently the internal metric I_{ij} is diagonal and

$$|I_{jj}| = \phi_*^2(x) \tag{8}$$

We call the metric g_{ab} on the bundle P compatible with the metric $g_{\mu\nu}$ and I_{ij} if

$$\begin{aligned} g_{ab} \hat{\xi}_\mu^a \hat{\xi}_\nu^b = g_{\mu\nu}, \qquad & g_{ab} \hat{\xi}_i^a \hat{\xi}_j^b = I_{ij} \\ g_{ab} \hat{\xi}_\mu^a \hat{\xi}_k^b = 0 \qquad & (a, b = 1 \rightarrow 4 + n) \end{aligned} \tag{9}$$

In the horizontal lift basis $(\hat{\xi}_i, \hat{\xi}_\mu)$, g_{ab} can be written as

$$g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & I_{ij} \end{pmatrix} \tag{10}$$

We see that the internal metric I_{ij} is a local metric; it depends on the coordinates of space-time.

4. THE CURVATURE OF THE BUNDLE *P*

The curvature tensors $R^i_{\mu\nu}$ of the bundle P are defined from the connection B^i_μ and local structural constants $F^i_{jk}(x)$; the $R^i_{\mu\nu}$ satisfy

$$[D_\mu, D_\nu] = -R^i_{\mu\nu}\hat{\xi}_i \tag{11}$$

$$R^i_{\mu\nu} = \partial_\mu B^i_\nu - \partial_\nu B^i_\mu + F^i_{jk} B^j_\mu B^k_\nu \tag{12}$$

$$\partial_i R^i_{\mu\nu} = -F^j_{ik} R^k_{\mu\nu} \tag{13}$$

Through serial calculating, we find connections and Ricci tensors as

$$\begin{aligned} &\hat{\Gamma}^i_{\mu\nu} = -\tfrac{1}{2} R^i_{\mu\nu}, \qquad \hat{\Gamma}^\nu_{i\mu} = \tfrac{1}{2} g^{\mu\rho} I_{ij} R^j_{\mu\rho} \\ &\hat{\Gamma}^\mu_{ij} = -\tfrac{1}{2} g^{\mu\rho}\partial_\rho I_{ij}, \qquad \hat{\Gamma}^i_{\mu j} = \tfrac{1}{2} I^{ik}\partial_\mu I_{jk} \\ &\hat{\Gamma}^k_{ij} = \tfrac{1}{2} F^m_{ni} I^{kn} I_{mj} + \tfrac{1}{2} F^m_{nj} I^{kn} I_{mi} + \tfrac{1}{2} F^k_{ij} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \hat{R}_{\mu\nu} = {} & R_{\mu\nu}(M) + \tfrac{1}{2} g^{\rho\lambda} I_{ik} R^i_{\rho\nu} R^k_{\mu\lambda} - \tfrac{1}{2}\partial_\nu (I^{ik}\partial_\mu I^{ik}) \\ & - \tfrac{1}{4} I^{ik} I^{jm} \partial_\nu I_{jk} \partial_\mu I_{im} + \tfrac{1}{2}\Gamma^\lambda_{\mu\nu} I^{ik}\partial_\lambda I_{ik} \\ \hat{R}_{ik} = {} & -\tfrac{1}{4} g^{\mu\rho} g^{\nu\lambda} I_{km} I_{in} R^m_{\nu\rho} R^n_{\mu\lambda} - \tfrac{1}{2}\Gamma^\mu_{\nu\mu} g^{\nu\rho}\partial_\rho I_{ik} \\ & + \tfrac{1}{4} g^{\mu\rho} I^{jm} (\partial_\mu I_{im}\partial_\rho I_{jk} + \partial_\mu I_{km}\partial_\rho I_{ij} - \partial_\mu I_{jm}\partial_\rho I_{ik}) \end{aligned} \tag{15}$$

Finally, we find the scale curvature of the bundle P,

$$\begin{aligned} \bar{R} = {} & R(M) - \tfrac{1}{4} g^{\mu\nu} g^{\rho\lambda} I_{ik} R^i_{\nu\rho} R^k_{\mu\lambda} - 2 g^{\mu\nu}\nabla_\nu \partial_\mu (-I)^{1/2}/(-I)^{1/2} \\ & + g^{\mu\nu}\partial_\mu (-I)^{1/2}\, \partial_\nu (-I)^{1/2}/(-I) + \tfrac{1}{4} g^{\mu\nu}\partial_\mu I^{ij}\partial_\nu I_{ij} \end{aligned} \tag{16}$$

where $I = \det(I_{ij})$.

5. THE ACTION INTEGRAL

The Einstein–Hilbert action integral S_{n+4} of the bundle space P is written as

$$S_{n+4} = \int (g)^{1/2}\, \bar{R}\, d^4x\, dV_n \tag{17}$$

for convenience we denote

$$S_{n+4} = S_1 + S_2 + S_3$$

where

$$S_1 = \int d^4x \, dV_n \, (g)^{1/2} \, R(M)$$

$$S_2 = \int d^4x \, dV_n \, (g)^{1/2} \, (-\tfrac{1}{4} g^{\mu\nu} g^{\rho\lambda} I_{ik} R^i_{\nu\rho} R^k_{\mu\lambda})$$

$$S_3 = \int d^4x \, dV_n \, (g)^{1/2} \, g^{\mu\nu} \partial_\mu (-I)^{1/2} \, \partial_\nu (-I)^{1/2} / (-I) + \tfrac{1}{4} \partial_\mu I^{ij} \partial_\nu I_{ij})$$

Taking the gauge potential B^i_μ as variable, we get the variation of the action S_{n+4} equal to the variation of the action S_2. From $\delta S_{n+4} = \delta S_2 = 0$, we get a generalized equation of the gauge field

$$\partial_\mu (g^{1/2} R^{\mu\nu}_k) = F^i_{km} B^m_\rho \, g^{1/2} R^{\rho\nu}_i \tag{18}$$

Taking the torsion $T^\lambda_{\mu\nu} = \frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})$ as variable, we have the field equation of the torsion from $\delta S_{n+4} = \delta S_1 = 0$,

$$T^\lambda_{\mu\nu} = \tfrac{1}{4} [\partial_\mu (-I)^{1/2} \, \delta^\lambda_\nu / (-I)^{1/2} - \partial_\nu (-I)^{1/2} \, \delta^\lambda_\mu / (-I)^{1/2}] \tag{19}$$

Formula (19) shows that the torsion of space-time stems from the local metric of internal space.

6. DISCUSSION AND CONCLUSION

We have established a new theory of the principal fiber bundle which is a higher-dimensional unification of the gauge theory and gravitation with torsion. Here the fiber bundle is provided with a local metric of internal space. Every fiber has a enlarged factor $\phi(x)$; the enlarged fibers can be depicted in an intuitive geometric way. If we assume that the fiber is one-dimensional, it can be seen in Fig. 1 that every fiber on M possesses a

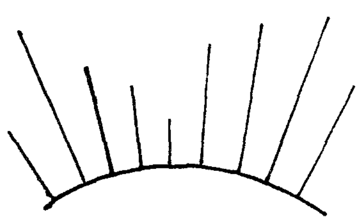

Fig. 1. Every fiber on M possesses a different length.

different length; in other words, the fibers have been drawn out like elastic and all the fibers are no longer the same. This is the difference between the bundle $P(M, G)$ and the bundle $P(M, F_x)$.

It must be pointed out that due to the above introduction of a mapping structure to the bundle P a local metric appears in internal space.

As we know the induced mapping $\phi_*(x)$ of $\phi(x)$ is a scale function of space-time, it can give a local vector field $\partial_\mu \phi_*(x)/\phi_*(x)$ at a local neighborhood of x. If we assume that the spin density vector S_μ of the gauge field determines the local vector $\partial_\mu \phi_*/\phi_*$ i.e.,

$$S_\mu = (\tfrac{3}{4}n/\kappa)\, \partial_\mu \phi_*/\phi_* \tag{20}$$

then from equations (19) and (20) we easily get the known Cartan equation

$$T_\mu = \kappa S_\mu \tag{21}$$

where κ is a constant. From here we see that our theory is an extended Cartan theory in a gauge field with spin.

In addition, if we integrate over all the internal space in the action (17), we get

$$S_{n+4} = V_n S_{\mathrm{BD}} + S_2 \tag{22}$$

where $S_{\mathrm{BD}} = \int dx^4\, (-g)^{1/2}\, (R\psi - \psi_{,\mu}\psi^{\mu}_{,}/\psi)$, and we have taken $\psi = (-I)^{1/2}$. Here S_{BD} is the Brans–Dicke action.

For the field equations of torsion we have $\delta S_{n+4} = \delta S_{\mathrm{BD}} = 0$. Therefore the Brans–Dicke theory (Rhamand *et al.*, 1988) with torsion is included in our theory.

Finally, we comment on the term S_3 in the action S_{n+4}. Using formulas (7) and (20), we rewrite S_3 as

$$S_3 = \int dx^4\, dV_n\, g^{1/2} 4\kappa^2 S^\mu S_\mu (1 + 1/n)/9 \tag{23}$$

The formula (23) shows that the term S_3 describes a spin self-interaction of the gauge field. In consideration of the physical significance of S_1 and S_2 the theory of principal fiber bundles can simultaneously describe gauge field gravitation with torsion and the self-interaction of the spin of the gauge field. We have found an important geometric object between the torsion and the spin of a gauge field. This geometric object is just the local metric of internal space.

REFERENCES

Cho, Y. M. (1975). *Journal of Mathematical Physics*, **16**(10), 2029.
Kalinowski, M. W. (1981). *International Journal of Theoretical Physics*, **20**, 563.
Kibble, T. W. B. (1961). *Journal of Mathematical Physics*, **2**, 212.
Madore, J. (1990). *Physical Review D*, **41**, 3709.
Ramanand, Jha., Lord, E. A., and Sinha, K. P. (1988). *General Relativity and Gravitation*, **20**(9), 943.
Sciama, D. W. (1962). In *Recent Developments in General Relativity*, Pergamon, New York.
Utiyama, R. (1956). *Physical Review*, **101**, 1597.
Von Westenholz, C. (1981). *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam.
Wu, T. T., and Yang, C. N. (1975). *Physical Review D*, **12**, 3845.
Wu, T. T., and Yang, C. N. (1976). *Nuclear Physics B*, **107**, 365.